

Disjunctive programming and relaxations of polyhedra

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Received: 27 April 2012 / Accepted: 16 January 2013 / Published online: 7 February 2013
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Abstract Given a polyhedron L with h facets, whose interior contains no integral points, and a polyhedron P , recent work in integer programming has focused on characterizing the convex hull of P minus the interior of L . We show that to obtain such a characterization it suffices to consider all relaxations of P defined by at most $n(h-1)$ among the inequalities defining P . This extends a result by Andersen, Cornuéjols, and Li.

Keywords Mixed integer programming · Disjunctive programming · Polyhedral relaxations

Mathematics Subject Classification (2000) 90C10 · 90C11 · 90C57 · 52B11

1 Introduction

Given polyhedra $P, L \subseteq \mathbb{R}^n$, we denote with

$$P \setminus L := \overline{\text{conv}(P - \text{int}L)}, \quad (1)$$

Supported by the Progetto di Eccellenza 2008-2009 of the Fondazione Cassa di Risparmio di Padova e Rovigo.

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where “ $\overline{\text{conv}}$ ” indicates the closed convex hull, “ $-$ ” the set difference, and “ int ” the topological interior. Let $Ax \leq b$ be a system of inequalities defining P . We denote by $\mathcal{R}^q(A, b)$ the family of the polyhedral relaxations of P that consist of the intersection of the half-spaces corresponding to at most q inequalities of the system $Ax \leq b$. In this note we prove the following theorem:

Theorem 1 *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and L be polyhedra in \mathbb{R}^n and let $h \geq 2$ be the number of facets of L . Then*

$$P \setminus L = \bigcap_{R \in \mathcal{R}^{n(h-1)}(A, b)} R \setminus L.$$

In the next section we provide a proof of this theorem, and we sketch a construction showing that the result does not hold if one considers polyhedra in $\mathcal{R}^{n(h-1)-1}(A, b)$. We now motivate it by providing an application to mixed integer programming.

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron and let $S = \mathbb{Z}^p \times \mathbb{R}^{n-p}$, for some p , $1 \leq p \leq n$. A mixed-integer set \mathcal{F} is a set of the form $\{x \in P \cap S\}$. Most of the research has focused on obtaining inequalities that are valid for \mathcal{F} , or equivalently, for $\text{conv}\mathcal{F}$, where “ conv ” indicates the convex hull. The operator defined in (1) was first considered in the mixed integer programming community by Andersen et al. [2], and it may be viewed as a special case of the disjunctive programming approach invented by Balas [3]. A convex set L is *S-free* if $\text{int}L$ does not contain any point in S . Given a mixed-integer set \mathcal{F} in the form described above and an S -free polyhedron L , \mathcal{F} is obviously contained in $P \setminus L$. It follows that any valid inequality for $P \setminus L$ is also valid for \mathcal{F} . The converse is also true: If P is a rational polyhedron and $ax \leq \beta$ is a valid inequality for \mathcal{F} , then $ax \leq \beta$ is valid for $P \setminus L$, for some S -free polyhedron L [13, 7]. This provides a motivation for the study of valid inequalities for $P \setminus L$ when L is a polyhedron, a setting that is receiving extensive interest from the community (see for example [4, 6, 10–13]).

Theorem 1 shows that in order to derive the inequalities that are essential in a description of $P \setminus L$, it is necessary and sufficient to consider inequalities that are valid for a relaxation of P comprising a number of inequalities that is a function of the dimension of the ambient space and of the number of facets of L .

Let $S = \mathbb{Z}^p \times \mathbb{R}^{n-p}$, for some p , $1 \leq p \leq n$. A *split* is a set L such that $L = \{x \in \mathbb{R}^n : \pi_0 \leq (\pi, 0)x \leq \pi_0 + 1\}$, for some $\pi \in \mathbb{Z}^p$, $\pi_0 \in \mathbb{Z}$. Clearly a split is an S -free convex set. Balas and Perregaard [5] prove Theorem 1 when P is contained in the unit cube and L is a split of the form $\{x \in \mathbb{R}^n : 0 \leq x_i \leq 1\}$, $1 \leq i \leq p$. Andersen et al. [1] prove Theorem 1 when L is a split, and they pose as an open question if their result generalizes to other polyhedra L . A shorter proof of the same result has been recently provided by Dash et al. [9], and uses the equivalence between split cuts and mixed-integer rounding (MIR) cuts. All these proofs do not seem to be extendable to a more general case.

Andersen et al. [1] also prove that, if L is a split in \mathbb{R}^n , in Theorem 1 it is enough to consider polyhedra in $\mathcal{R}^n(A, b)$ defined by linearly independent inequalities. Furthermore they show that if L is defined by only two inequalities, one cannot generally restrict to polyhedra in $\mathcal{R}^n(A, b)$ defined by linearly independent inequalities.

2 Proof of main result

The following lemma is well-known, as it is an equivalent formulation of Carathéodory's theorem (see for example [14]).

Lemma 1 *Let G be a matrix of size $m \times d$ and let \bar{r} be an extreme ray of the cone $\{r \in \mathbb{R}^m : r \geq 0, rG = 0\}$. Then \bar{r} has at most $d + 1$ positive components.*

Corollary 1 *Let $A^i, i = 1, \dots, k$ be $m^i \times n$ matrices and let $b^i, i = 1, \dots, k$ be vectors of dimension m^i . Let $(\bar{r}^i \in \mathbb{R}^{m^i}, \bar{s}^i \in \mathbb{R} : i = 1, \dots, k)$ be an extreme ray of the cone defined by the system*

$$\begin{aligned} -r^1 A^1 + r^i A^i &= 0 & i = 2, \dots, k \\ r^1 b^1 - r^i b^i + s^1 - s^i &= 0 & i = 2, \dots, k \\ r^i &\geq 0 & i = 1, \dots, k \\ s^i &\geq 0 & i = 1, \dots, k. \end{aligned}$$

Then $(\bar{r}^i, \bar{s}^i : i = 1, \dots, k)$ has at most $n(k - 1) + k$ positive components.

Proof The system

$$\begin{aligned} -r^1 A^1 + r^i A^i &= 0 & i = 2, \dots, k \\ r^1 b^1 - r^i b^i + s^1 - s^i &= 0 & i = 2, \dots, k \end{aligned}$$

comprises of $(n + 1)(k - 1)$ equations. By Lemma 1, $(\bar{r}^i, \bar{s}^i : i = 1, \dots, k)$ has at most $(n + 1)(k - 1) + 1 = n(k - 1) + k$ positive components. \square

(In the above proof, if $k = 1$ we intend the set of indices $i = 2, \dots, k$ to be empty.)

For $i = 1, \dots, k$ consider polyhedra $P^i = \{x \in \mathbb{R}^n : A^i x \leq b^i\}$ and cones $C^i := \{x \in \mathbb{R}^n : A^i x \leq 0\}$. So C^i is the recession cone of P^i if P^i is nonempty. By Minkowski-Weil's theorem (see for example [14]) there exist polytopes Q^i , for $i = 1, \dots, k$, such that

$$P^i = Q^i + C^i, \quad i = 1, \dots, k,$$

where $P^i = \emptyset$ if and only if $Q^i = \emptyset$. Let

$$\tilde{P} := \text{conv} \bigcup_{i=1}^k Q^i + \text{cone} \bigcup_{i=1}^k C^i, \quad (2)$$

where “cone” denotes the conic hull. Again, $\tilde{P} = \emptyset$ if and only if $\bigcup_{i=1}^k Q^i = \emptyset$.

Let S' be the following system of inequalities:

$$A^i x^i - b^i \lambda^i \leq 0 \quad i = 1, \dots, k \quad (3)$$

$$x - \sum_{i=1}^k x^i = 0 \quad (4)$$

$$\sum_{i=1}^k \lambda^i = 1 \quad (5)$$

$$\lambda^i \geq 0 \quad i = 1, \dots, k. \quad (6)$$

Given a polyhedron $P = \{(x, y) \in \mathbb{R}^{n+d} : Ax + Gy \leq b\}$, we denote with $\text{proj}_x P \subseteq \mathbb{R}^n$ the orthogonal projection of P onto the space of the x -variables. More precisely $\text{proj}_x P := \{x \in \mathbb{R}^n, \exists y \in \mathbb{R}^d : Ax + Gy \leq b\}$. The following theorem is similar to Balas' theorem on union of polyhedra [3].

Theorem 2 [8] *Given k polyhedra $P^i = \{x \in \mathbb{R}^n : A^i x \leq b^i\} = Q^i + C^i$, let \tilde{P} defined as in (2), and let $Y' \subset \mathbb{R}^{n+(n+1)k}$ be the polyhedron defined by the system (3)–(6). Then $\tilde{P} = \text{proj}_x Y'$.*

Furthermore, if either $P^i = \emptyset, i = 1, \dots, k$, or if $P^i \neq \emptyset, i = 1, \dots, k$, then $\tilde{P} = \overline{\text{conv}} \bigcup_{i=1}^k P^i$.

We now prove Theorem 1.

Proof Clearly $P \setminus L \subseteq \bigcap_{R \in \mathcal{R}^{n(h-1)}(A,b)} R \setminus L$, thus we need to show the reverse inclusion.

Every inequality in the system $Ax \leq b$ is valid for some $R \in \mathcal{R}^1(A, b)$. Since $h \geq 2$, $R \in \mathcal{R}^{n(h-1)}(A, b)$ and therefore $P \supseteq \bigcap_{R \in \mathcal{R}^{n(h-1)}(A,b)} R \setminus L$.

If L is not full-dimensional, $\text{int} L = \emptyset$, $P \setminus L = P \supseteq \bigcap_{R \in \mathcal{R}^{n(h-1)}(A,b)} R \setminus L$, and the theorem follows. So we assume that L is a full-dimensional polyhedron with h facets. Hence $L = \{x \in \mathbb{R}^n : c^i x \leq \delta^i, i = 1, \dots, h\}$, where each inequality $c^i x \leq \delta^i$ defines a facet of L .

For $i = 1, \dots, h$, let $A^i x \leq b^i$ be the system obtained from $Ax \leq b$ by adding inequality $-c^i x \leq -\delta^i$ and let $P^i := \{x \in \mathbb{R}^n : A^i x \leq b^i\}$. Let k be defined as follows. If $P^i = \emptyset$ for every $i = 1, \dots, h$, let $k = h$. Otherwise let $k \geq 1$ be the number of nonempty polyhedra among $P^i, i = 1, \dots, h$, and we assume that the nonempty polyhedra are P^1, \dots, P^k . It follows from the definition of $P \setminus L$ that

$$P \setminus L = \overline{\text{conv}} \bigcup_{i=1}^k P^i.$$

Let S be the following system, obtained from (3)–(6) by using Eqs. (4) and (5) to eliminate vector x^1 and scalar λ^1 :

$$\begin{aligned}
A^1 x - A^1 \sum_{i=2}^k x^i + b^1 \sum_{i=2}^k \lambda^i &\leq b^1 \\
A^i x^i - b^i \lambda^i &\leq 0 \quad i = 2, \dots, k \\
\sum_{i=2}^k \lambda^i &\leq 1 \\
\lambda^i &\geq 0 \quad i = 2, \dots, k.
\end{aligned}$$

Let Y be the polyhedron defined by S . Note that Y is a polyhedron in $\mathbb{R}^{n+(n+1)(k-1)}$ involving vectors x, x^2, \dots, x^k and scalars $\lambda^2, \dots, \lambda^k$. Furthermore Theorem 2 implies that

$$P \setminus L = \text{proj}_x Y.$$

Let U be the set of the extreme rays $(r^i, s^i : i = 1, \dots, k)$ of the cone defined by the system

$$-r^1 A^1 + r^i A^i = 0 \quad i = 2, \dots, k \quad (7)$$

$$r^1 b^1 - r^i b^i + s^1 - s^i = 0 \quad i = 2, \dots, k \quad (8)$$

$$r^i \geq 0 \quad i = 1, \dots, k \quad (9)$$

$$s^i \geq 0 \quad i = 1, \dots, k. \quad (10)$$

Since $P \setminus L = \text{proj}_x Y$, it is well-known that

$$P \setminus L = \{x \in \mathbb{R}^n : r^1 A^1 x \leq r^1 b^1 + s^1, \forall (r^i, s^i : i = 1, \dots, k) \in U\}. \quad (11)$$

Let $(\bar{r}^i, \bar{s}^i : i = 1, \dots, k)$ be a ray in U , and let $ax \leq \beta$ be the corresponding valid inequality for $P \setminus L$, where $a = \bar{r}^1 A^1, \beta = \bar{r}^1 b^1 + \bar{s}^1$. To prove $P \setminus L \supseteq \bigcap_{R \in \mathcal{R}^{n(h-1)}(A, b)} R \setminus L$, it suffices to show that there exists a polyhedron $\bar{R} \in \mathcal{R}^{n(h-1)}(A, b)$ such that $ax \leq \beta$ is valid for $\bar{R} \setminus L$. Since $P \supseteq \bigcap_{R \in \mathcal{R}^{n(h-1)}(A, b)} R \setminus L$, we assume that the inequality $ax \leq \beta$ is not valid for P . We now construct a polyhedron $\bar{R} \in \mathcal{R}^{n(h-1)}(A, b)$ such that $ax \leq \beta$ is valid for $\bar{R} \setminus L$.

For $i = 1, \dots, k$, let R^i be the polyhedron defined by the inequalities in $Ax \leq b$ corresponding to positive components of \bar{r}^i .

Note that when $k < h$, by definition of k , $P \neq \emptyset$ and for $i = k+1, \dots, h$, $P^i = P \cap \{x \in \mathbb{R}^n : c^i x \geq \delta^i\} = \emptyset$. Since $P \neq \emptyset$, it follows by Carathéodory's theorem (see for example [14]) that, for $i = k+1, \dots, h$, there exist a polyhedron R^i defined by at most n linearly independent inequalities in $Ax \leq b$ such that $R^i \cap \{x \in \mathbb{R}^n : c^i x \geq \delta^i\} = \emptyset$.

We now show that for $i = 1, \dots, h$, inequality $ax \leq \beta$ is valid for $R^i \cap \{x \in \mathbb{R}^n : c^i x \geq \delta^i\}$. For $i = 1, \dots, k$, by (7)–(11) we have that $a = \bar{r}^i A^i, \beta = \bar{r}^i b^i + \bar{s}^i$, and $\bar{r}^i, \bar{s}^i \geq 0$, thus $ax \leq \beta$ is valid for $R^i \cap \{x \in \mathbb{R}^n : c^i x \geq \delta^i\}$. Moreover for $i = k+1, \dots, h$, $ax \leq \beta$ is valid for $R^i \cap \{x \in \mathbb{R}^n : c^i x \geq \delta^i\} = \emptyset$. Now

let $\bar{R} = \bigcap_{i=1}^h R^i$. Hence $ax \leq \beta$ is valid for $\bar{R} \cap \{x \in \mathbb{R}^n : c^i x \geq \delta^i\}$ for every $i = 1, \dots, h$. This shows that $ax \leq \beta$ is valid for $\bar{R} \setminus L$.

We finally show $\bar{R} \in \mathcal{R}^{n(h-1)}(A, b)$. For $i = 1, \dots, k$, since $ax \leq \beta$ is not valid for P and $P \subseteq R^i$, $ax \leq \beta$ is not valid for R^i . Since by (7)–(11) we have that $a = \bar{r}^i A^i$, $\beta = \bar{r}^i b^i + \bar{s}^i$, and $\bar{r}^i, \bar{s}^i \geq 0$, it follows that the component of \bar{r}^i corresponding to $c^i x \geq \delta^i$ must be positive. By Corollary 1 the positive components of the vector $(\bar{r}^i : i = 1, \dots, k)$ are at most $n(k-1) + k$, and by the previous argument, the k components of $(\bar{r}^i : i = 1, \dots, k)$ corresponding to the inequalities $c^i x \geq \delta^i, i = 1, \dots, k$, are all positive. This shows that $\bigcap_{i=1}^k R^i$ is defined by at most $n(k-1)$ inequalities of $Ax \leq b$. Moreover for $i = k+1, \dots, h$, R^i is defined by at most n inequalities of $Ax \leq b$. It follows that \bar{R} is defined by at most $n(k-1) + n(h-k) = n(h-1)$ inequalities of $Ax \leq b$, hence $\bar{R} \in \mathcal{R}^{n(h-1)}(A, b)$. \square

We conclude this paper showing that the bound given in Theorem 1 is tight. For $n = 1$ the result is trivial since L has at most 2 facets, so assume $n \geq 2$. For every $n \geq 2$ and $h \geq 2$, we sketch the construction of a polyhedron P in \mathbb{R}^n and a polyhedron L with h facets such that

$$P \setminus L \subset \bigcap_{R \in \mathcal{R}^{n(h-1)-1}(A, b)} R \setminus L.$$

Figure 1 illustrates the construction for $n = 2, h = 3$.

Let $L' = \{x \in \mathbb{R}^n : c^i x \leq \delta^i, i = 1, \dots, h\}$ be a full dimensional polyhedron, where inequalities $c^i x \leq \delta^i$ are in one to one correspondence with the $h \geq 2$ facets F^i of L' . For every $i = 1, \dots, h$, let f^i be a point in the relative interior of F^i . Let $\epsilon > 0$ be such that for every $i = 1, \dots, h$

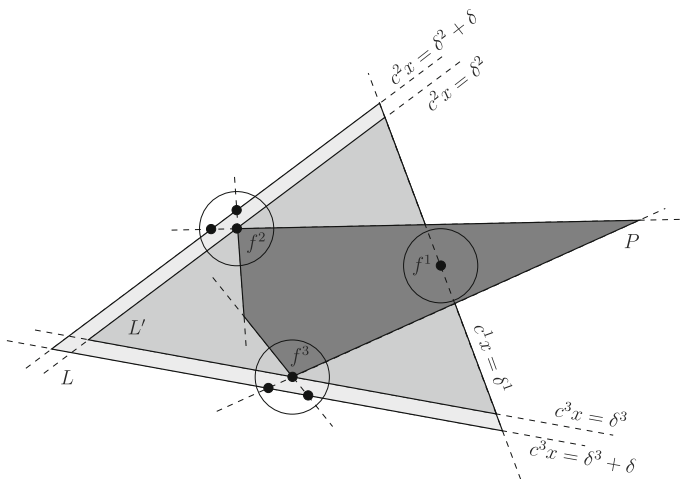


Fig. 1 Construction for $n = 2, h = 3$

- i) the strict inequalities $c^j x < \delta^j$ are valid for $f^i + \epsilon B$, for $j = 1, \dots, h$ with $j \neq i$, where B is the unit ball in \mathbb{R}^n .

For every $i = 2, \dots, h$, let $A^i x \leq b^i$ be a system of n linearly independent inequalities, such that:

- ii) $A^i f^i = b^i$,
 iii) $c^i x \leq \delta^i$ is valid for $R^i := \{x \in \mathbb{R}^n : A^i x \leq b^i\}$, and $R^i \cap \{x \in \mathbb{R}^n : c^i x = \delta^i\} = f^i$,
 iv) $f^j + \epsilon B \subseteq R^i$, for $j = 1, \dots, h$ with $j \neq i$.

(The existence of such systems follows from the definition of f^i , $i = 1, \dots, h$, and by i)). For $i = 2, \dots, h$ and $j = 1, \dots, n$, let $a^{ij} x \leq \beta^{ij}$ be the j th inequality of the system $A^i x \leq b^i$, and let $A^{ij} x \leq b^{ij}$ be the system obtained from $A^i x \leq b^i$ by removing $a^{ij} x \leq \beta^{ij}$.

Since for $i = 2, \dots, h$, the polyhedra R^i are translate of polyhedral cones and by ii) R^i has apex f^i , it follows from iii) that for every $i = 2, \dots, h$, $j = 1, \dots, n$, and $\delta > 0$, there exists a unique point x^{ij} that satisfies

- v) $A^{ij} x^{ij} = b^{ij}$ and $c^i x^{ij} = \delta^i + \delta$.

Let $\delta > 0$ be small enough such that $x^{ij} \in f^i + \epsilon B$ for every $i = 2, \dots, h$ and $j = 1, \dots, n$.

Let $L := \{x \in \mathbb{R}^n : c^1 x \leq \delta^1, c^i x \leq \delta^i + \delta, i = 2, \dots, h\}$ and let $P = \bigcap_{i=2}^h R^i$. Note that P is defined by the system $Ax \leq b$ consisting of all inequalities in systems $A^i x \leq b^i$, $i = 2, \dots, h$. Since by iii), for $i = 2, \dots, h$, inequalities $c^i x \leq \delta^i$ are valid for P and $\delta > 0$, then $P \cap \{x \in \mathbb{R}^n : c^i x \geq \delta^i + \delta\} = \emptyset$ for every $i = 2, \dots, h$. This shows that $P \setminus L = P \cap \{x \in \mathbb{R}^n : c^1 x \geq \delta^1\}$. Since by i), $c^1 f^2 < \delta^1$ and by ii), iv), $f^2 \in P$, the inequality $c^1 x \geq \delta^1$ is not valid for P , and so $c^1 x \geq \delta^1$ is redundant for the system defining $P \setminus L$.

We now show that for every $R \in \mathcal{R}^{n(h-1)-1}(A, b)$, the inequality $c^1 x \geq \delta^1$ is not valid for $R \setminus L$.

Let $R \in \mathcal{R}^{n(h-1)-1}(A, b)$. Since the system $Ax \leq b$ contains $n(h-1)$ inequalities, R contains the polyhedron defined by the system $Ax \leq b$ deprived of a single inequality. We assume without loss of generality that this inequality is $a^{21} x \leq \beta^{21}$, and so is the first inequality of the system $A^2 x \leq b^2$. By v), the point x^{21} is such that $A^{21} x^{21} = b^{21}$ and $c^2 x^{21} = \delta^2 + \delta$. By the choice of δ , $x^{21} \in f^2 + \epsilon B$, so it follows by iv) that $x^{21} \in R^i$ for every $i = 3, \dots, h$. Hence $x^{21} \in R$.

Since $c^2 x^{21} = \delta^2 + \delta$, and $c^2 x \leq \delta^2 + \delta$ is valid for L , x^{21} does not belong to the interior of L . This shows that x^{21} belongs to $R \setminus L$. Since x^{21} belongs to $f^2 + \epsilon B$, then by i), $c^1 x^{21} < \delta^1$. Hence $c^1 x \geq \delta^1$ is not valid for $R \setminus L$.

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